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Exploring Mathematical Reasoning and Proof with Indonesian Senior High School Students

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Abstract

Constructing mathematical reasoning and proof skills is challenging for all students. In Indonesia, students' abilities to apply reasoning and proof skills are closely observed under the Ministry of Education's new curriculum. Nevertheless, students' efforts to develop reasoning and proof may require more feedback on various mathematical tasks for the implementation of the new curriculum. Therefore, this study aims to investigate the extent to which students demonstrate reasoning and proof in solving mathematical questions. A mixed-methods approach was employed, using an embedded research designs to collect quantitative and qualitative data. This research employed simple random sampling to collect quantitative data from 384 tenth grade students through test questions. Six scripts were chosen for in-depth examination in the qualitative analysis. The findings revealed that students struggled to demonstrate reasoning and proof; they relied on validation through counterexamples to support their assertions, failing to establish relationships between premises and conclusions. Students' lack of understanding of underlying concepts resulted in low performance on reasoning and proof questions. From the findings, we suggest that teachers become aware of assigning tasks and exposing students to exercises related to reasoning and proof, to enhance their mathematical reasoning and proof abilities.

Keywords: mathematical reasoning and proof; conjecture; argument.

1 Introduction

Curriculum designers have acknowledged the educational significance of reasoning and proof. The National Council of Teachers of Mathematics has combined reasoning and proof into a single process standard rather than treating them as separate concepts. They firmly believe that reasoning and proof should be a fundamental and essential part of mathematics education for all students from prekindergarten to Grade 12 [32] . Furthermore, reasoning and proof have become increasingly significant in mathematics and the acquisition of mathematical knowledge, as noted by Cai and Cirillo [9]. Hence, proficiency in reasoning and proof is a fundamental skill that mathematics students must acquire.

According to the results of the Programme for International Student Assessment (PISA), Indonesian students still demonstrate limited reasoning skills. The results were concerning when the knowledge was assessed in mathematical literacy aspects. The Organisation for Economic Co-operation and Development (OECD) defines mathematical literacy as the ability to construct, apply, and comprehend mathematics in various circumstances, including mathematical reasoning. On this matter, Indonesian students were ranked 70^{th} in mathematical literacy with 366 points in the PISA 2022 results [36].

Numerous studies indicate that students have significant challenges with reasoning and proof. Students should be provided with opportunities to enhance their ability to explore reasoning and proof [44]. Among the opportunities are appropriate intervention to ease students' understanding [43], enhancing their ability to make connections within proofs [19], and developing their proof-related competences [20]. In addition, research has found that many secondary school and college students struggle with comprehending reasoning and proofs [47], particularly Indonesian students [2, 3], Indonesian senior high school students. Herizal et al. [21] found that students need more assistance in engaging with mathematical proof.

Most literature focuses on reasoning and proof in congruent triangles in geometry and the identity of trigonometry [6, 33]. Researchers have emphasised the crucial role of reasoning and proof in geometry and other subjects at different grade levels through instructional practices [41] and teachers' conceptions of proof [27]. There is a significant research deficit as the focus has only been on these mathematical areas. Despite the seriousness highlighted in terms of epistemological and didactic obstacles [5], where examples used unintentionally do not enhance conceptual understanding, awareness of getting students to improve may need more attention in specific contexts, such as applying mathematics to physics subjects [48].

This underscores the need for future research to incorporate reasoning and proof into all areas of the mathematics curriculum. It is crucial to investigate students' obstacles and struggles in demonstrating reasoning and proof when solving mathematical problems. Despite the recognition of these skills as necessary, a comprehensive understanding of how well students demonstrate reasoning and proof across various mathematical tasks is lacking. This study aims to investigate the extent to which students demonstrate reasoning and proof in solving various questions. The research questions are:

(1) What are students' achievement in reasoning and proof?

(2) To what extent do students demonstrate reasoning and proof in solving mathematical problems?

This research seeks to enhance understanding of how students use reasoning and proof in different mathematical scenarios, offering educators and curriculum developers valuable insights.

2 Related Works

Mathematical proof involves deriving conclusions through logical reasoning from fundamental notions and assumptions [49] . "For mathematicians, proof varies according to the discipline involved, although one essential principle underlies all its varieties: To specify the assumptions made and to provide an appropriate argument supported by valid reasoning to draw necessary conclusions" [16]. Mathematical reasoning is the logical process of deriving conclusions from facts or premises with a high level of deductive validity [35]. Proof is at the heart of mathematical reasoning [47, 24]. According to the studies above, mathematical reasoning and proof are learning activities essential to proving competence, such as making or investigating a conjecture, developing or evaluating an argument, and correcting or identifying a mistake [47]. This definition is relevant to the setting of the present study.

Reasoning and proof play a significant role in mathematics. A substantial number of mathematics educators argue that teaching students mathematical practices associated with proof is one of the most essential purposes of mathematics education since proof is one of the most critical components of mathematical practice [11]. Many mathematics educators maintain that proof is one of the pillars of mathematical practice and that training students in mathematical activities related to reasoning and proof is an essential goal of mathematics education [11, 40]. Similarly, reasoning is a crucial mathematical skill for students nationally and internationally. Mathematical reasoning is an essential topic in mathematics education [22]. When students fail to develop their mathematical reasoning, they lack an understanding of the essence of mathematical learning, reducing it to following a sequence of procedures and imitating examples [37]. Therefore, fostering reasoning and proof skills is imperative for the mathematics education of students across all age groups [42].

Moreover, developing students' mathematical reasoning and proof skills is a goal of several curricula, including the Indonesian curriculum. Specifically, reasoning and proof skills are a key process components for mathematics subjects outlined in the learning achievements at early childhood, primary education level, and secondary education level within the Merdeka Curriculum in Indonesia [25].

Despite being taught proof in mathematics classes, students performed poorly on the reasoning and proof assignments. Moreover, while the importance of proof in mathematics is well recognised, students often struggle with these skills. The challenges students faced with proofs at all educational levels and across national borders are extensively documented. These difficulties manifest in students' struggle with drawing conclusions and making conjectures [2], learning abstract algebra [3], and establishing proof when identifying patterns of statements [21], which has prompted educators to explore innovative teaching methods, such as developing reasoning and proof assessment instruments [38].

Based on the literature, the challenges encountered by secondary school students concerning reasoning and proofs include a lack of clarity regarding the meaning or objective of the proof, insufficient understanding of the concepts within the field of study, and unfamiliarity with specific proof strategies [15]. Furthermore, many students incorrectly belief that confirmatory examples alone can establish mathematical generalisation, relying on limited examples to support their claims [13]. Similarly, a study by [39] found that students frequently erroneously prove general statements using specific examples. Globally, researchers observe students routinely accepting examples as proof while overlooking the limitations of this reasoning [43, 8].

Furthermore, a key difficulty students face in reasoning and proof is their inability to establish relationships between given premises and conclusion [49]. The logical reasoning structure in a proof involves making a sequence of deductions that are assumed to be in line with the laws of logic to demonstrate the intended conclusion from the provided premises. Effective reasoning structuring involves arranging a student's deductions in a suitable sequence to demonstrate the intended conclusion based on the provided premises [50].

One common student error in demonstrating reasoning and proof is omitting the final step. Despite being able to articulate their reasoning verbally, they failed to demonstrate it in writing [50], and they require additional assistance as they struggled to create a proof scheme during the proofing steps [28]. They were routinely incomplete in the concluding process from a deductive perspective and tended to assume that the conclusion was self-evident. In addition, their study [14] found that none of the students utilised formal mathematical language in their reasoning and proof demonstrations. Nevertheless, all students endeavoured to justify their arguments and convince others of their accuracy. In generating formal proofs, students should integrate informal language into formal language, comprehend mathematical definitions, apply theorems, and establish links between mathematical objects. It is recommended that these skills be cultivated early, as pupils can exhibit reasoning abilities [14], despite variations in teaching methodologies throughout educational levels [10]. Most importantly, students must develop perseverance in learning [4]. Indonesia is revising the curriculum and emphasising reasoning and proof as learning objectives to enhance students' abilities in these areas. Therefore, investigating Indonesian students' abilities to develop reasoning and construct proofs when solving problems is essential.

3 Materials and Methods

3.1 Reserach design

The study examined Indonesian secondary 10^{th} grade students' ability to demonstrate reasoning and proof when solving mathematical problems. Therefore, the research employed a mixed approach, focusing on embedded research designs to collect quantitative and qualitative data. Quantitative data were collected through a large-scale test completed by 384 students. For qualitative data, the embedded design analyse specifics mathematics problems solved by participants. Six scripts containing mathematical problem solution were analysed for reasoning and proof demonstration. This triangulated data collected using the same instrument. It provides a better understanding of the survey results through qualitative analysis, namely document analysis.

3.2 Sampling

This study involved high school Indonesian students as participants. Indonesian students have demonstrated consistent difficulties in mathematics, as evidenced by recent studies. Mathematical proficiency has decline further due to reduced face to face instruction during the COVID-19 pandemic. The lack of interaction between educators and students has impeded the students' capacity for creativity, as has been proved in recent studies [46, 1]. Intervention are needed to prevent further deterioration, as recent findings indicate [34]. Therefore, monitoring high school students and creating opportunities to enhance their mathematical skills is essential. The assessment of their reasoning abilities as an important component in the curriculum needs more up-to-date input for improvement [30]. This research employed simple random sampling to collect data. We randomly selected 384 secondary school students from the Grade 10 population in Lombok, Indonesia. The population represented diverse backgrounds across school types (rural and urban) and cultures [23]. Additionally, all participants had experience solving mathematics problems requiring reasoning. In this study, the participants completed a set of test questions. After collecting data from the 384 students for the quantitative analysis, six scripts were chosen for in-depth examination in the qualitative analysis. This allows detailed investigation of Grade 10 students' reasoning and proof strategies.

3.3 Instrument

Aligning with the educational aims in Indonesia, students' reasoning and problem-solving abilities are closely monitored. These findings should inform interventions to improve students' mathematical reasoning. This study investigates higher secondary school students' reasoning abilities. It supports the educational policy of promoting essential skills like critical thinking, numeracy, and problem-solving, which involve the ability to think critically and analyse information with reasoning. The Merdeka Curriculum focuses on developing these skills in Grade 10 students, while the Minimum Competency Assessment evaluates them in Grades 5, 8, and 11 [31].

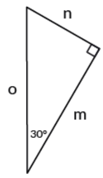
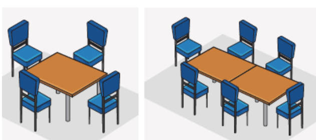
Therefore, this study adapted mathematics items on reasoning and proof from the Indonesian Mathematics Textbooks question [45], which is conducted under the Merdeka Curriculum. These reasoning and proof questions were based on the [47] category and consisted of making a conjecture, investigating a conjecture, developing an argument, evaluating an argument, and correcting a mistake. The content validity of this instrument was established by three experts from university and secondary school mathematics teachers. They validated the instrument for content construction and language appropriateness. The percentage of agreement of the expert is calculated using Borich's percentage agreement equation [7], namely,

$$\text{percentage of agreement} = \left[1 - \frac{A-B}{A+B}\right] \times 100\%,$$

where A indicates the highest score given by the validator, and B indicates the lowest score given by the validator. The agreement met the requirement of above 75%. Table 1 illustrates samples of the items.

Reasoning and proof ability data were collected through paper and pencil tests. The test consists of 11 items, namely making a conjecture, investigating a conjecture, developing an argument, evaluating an argument, and correcting a mistake. Student responses were analysed according to the study objectives. The analysis examined both overall student performance and specific errors per question.

Table 1: Samples of the items.

English version
Is the form $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ true? Explain your answer!
Putri named the sides of the triangle as below,  The opposite is the m side. The adjacent is the n side. The hypotenuse is o side. Writing down suggestions for Putri to improve her understanding! In your suggestions, make sure there is an explanation!
In the two pictures below, compare the number of tables and chairs. A rectangle table with four chairs is available. When two tables are joined together, they can hold six chairs, as illustrated in the picture below.  How many tables are required if 20 persons will eat together at one table? How did you find out? Explain your answer!

4 Results

The findings are presented based on the two research questions. The first research question aims to analyse using descriptive statistics, while the second uses document analysis for qualitative

4.1 Findings for research question one

Research Question One: What are students' achievement in reasoning and proof?

Table 2 shows the mean score achieved in the reasoning and proof test among the 384 students involved in the study.

Table 2: Reasoning and proof test score.

	N	**Mean**	**Std. Deviation**	**Percentage**
Reasoning and proof test	384	9.3854	4.1452	21.33%

Table 2 above depicts that the mean score of the 384 students is low at 9.3854 (SD = 4.1452), showing a percentage score of $21.33 \left(\frac{9.3854}{44} \times 100\right)$. This analysis indicates the upper-secondary school students obtained low-level achievement in reasoning and proof. The fact that the low level of students' mathematics achievement and mastery is undeniable. The majority of students often memorize mathematical formulas without comprehending their underlying concepts

Table 3 below shows the combines of the total marks, mean, standard deviation of reasoning and proof knowledge levels in different topics (exponents and logarithms, sequences and series, vector, trigonometry, and probability).

Table 3: Student achievement in reasoning and proof questions based on chapter.

Reasoning and proof					
Topic	**Exponents and logarithms**	**Sequence and series**	**Vector**	**Trigonometry**	**Probability**
Total marks	12	4	4	20	4
Mean	2.36	1.68	1.18	3.72	0.46
Average of percentage	19.67%	42%	29.50%	18.6%	11.5%
Std. deviation	1.64	0.94	0.96	2.54	0.87

The acquisition levels of reasoning and proof in specific topics which are demonstrated in descending levels are sequence and series (42%), vector (29.5%), exponents and logarithms (19.6%), trigonometry (18.6%), and probability (11.5%). Hence, probability and trigonometry show a lower achievement compared to other topics.

Table 4 shows acquisition levels of reasoning and proof in specific category which are demonstrated in descending levels make a conjecture (27.125%), investigating a conjecture (26.125%), developing an argument (25%), correcting a mistake (24.25%), evaluate an argument (10%) and principle of proof (10%).

Table 4: Student achievement in reasoning and proof questions based on category.

Category	**Make a conjecture**	**Develop an argument**	**Evaluate an argument**	**Investigate a conjecture**	**Correct a mistake**	**Principle of proof**
Total marks	8	8	8	8	8	4
Mean	2.17	2.00	0.80	2.09	1.94	0.40
Percentage	27.125	25	10	26.125	24.25	10
Std. deviation	1.31	1.28	1.29	1.69	1.67	0.72

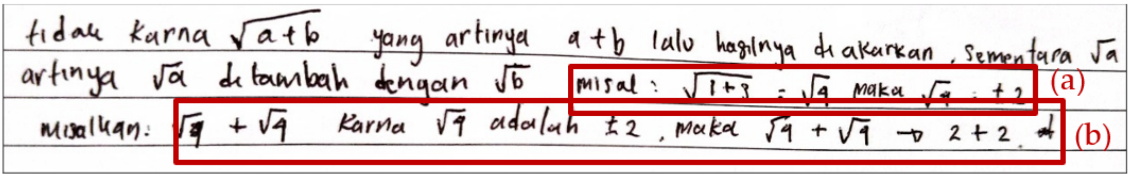
4.2 Findings for research question two

Research Question Two: To what extent do students demonstrate reasoning and proof in solving mathematical problems?

The following analyses aim to explore students' ability to demonstrate reasoning and proof in various aspects, including making a conjecture, investigating a conjecture, developing an argument, correcting a mistake, evaluating an argument and the principle of proof.

Question 1: Is the form $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ true? Explain your answer!

This question aims to test the ability to investigate a conjecture. For this question, students were assigned to investigate whether the assertion given is true or false, and they needed to provide a rationale by understanding the root form operation concept.
The following results were presented from the selected respondents' work.

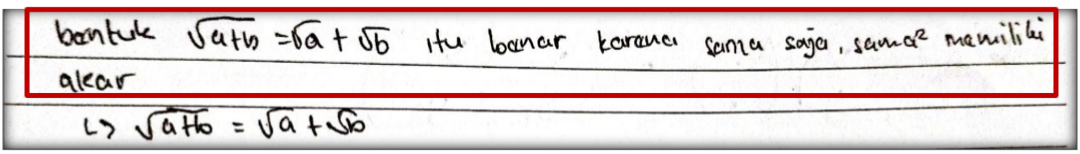


tidak karna $\sqrt{a+b}$ yang artinya $a+b$ lalu hasilnya di akarkan, Sementara $\sqrt{a}$ artinya $\sqrt{a}$ di tambah dengan $\sqrt{b}$ misal: $\sqrt{1+3} = \sqrt{4}$ maka $\sqrt{4} = \pm 2$ (a)
misalkan: $\sqrt{4} + \sqrt{4}$ karna $\sqrt{4}$ adalah ± 2, maka $\sqrt{4} + \sqrt{4} \to 2 + 2$ (b)

Remark: $\sqrt{a+b}$ means $a+b$ while it is added with $\sqrt{b}$, example $\sqrt{1+3} = \sqrt{4}$ thus $\sqrt{4} = \pm 2$
Example: $\sqrt{4} + \sqrt{4}$ because $\sqrt{4}$ is ± 2, thus $\sqrt{4} + \sqrt{4} \to 2 + 2$.

Figure 1: Respondent 1's solution for question 1.

Figure 1 depicts Respondent 1's (R1's) response to the questions using an empirical argument and providing little explanation, namely applying a counterexample. R1 operated using specified integers a and b, assuming that $a = 1$ and $b = 3$. R1 determined the value of $\sqrt{4}$ as shown in (a), compared to the value of $\sqrt{4} + \sqrt{4}$ as shown in (b), and discovered that they were not equal. This led to the conclusion that the assertion was false. The student's conclusion was accurate. However, the strategy to evaluate the assertion's truth was not convincing enough. Therefore, the student displayed difficulties in demonstrating reasoning and proof



bentuk $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ itu benar karena sama saja, sama² memiliki akar
↳ $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Remark: $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is real because same, same roots.

Figure 2: Respondent 4's solution for question 1.

Reviewing the attached response (shown in Figure 2) from R4 shows that the respondemt deduced directly that the form $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is true, as both possess common roots. The respondent's answer showed that the respondent has no basis for constructing proofs and demonstrating reasoning as he demonstrated a lack of comprehension about root-form operations. The students failed to identify the square root of a function using the distributive property, namely $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ They thought this property could be applied in any operation, including addition, namely
$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ without awareness that the distributive property holds in multiplication and division but not addition.

In conclusion, students encountered challenges demonstrating reasoning and proof when assigned to investigate a conjecture question. The fundamental cause of this problem is a deficiency in acquiring a thorough comprehension of specific mathematical concepts associated with the conjecture under consideration. During their investigation, students must address these concepts. An important finding in the analysis was the frequent use of a particular example to validate the accuracy of the presented assertion. This inclination indicated a possible deficiency in students' comprehension, as they can be prone to making generalisations based on a single example.

Question 2: Putri named the sides of the triangle in Figure 3 as follows,

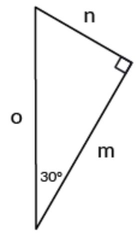
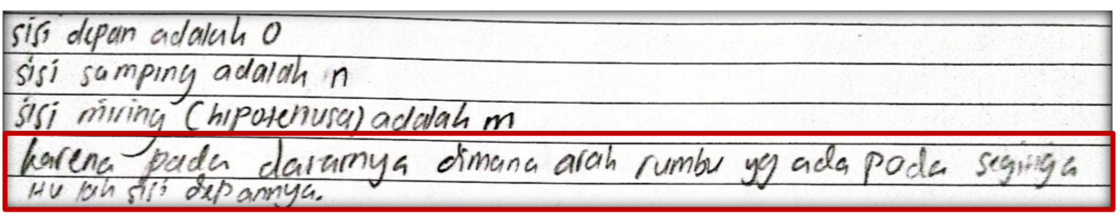


Figure 3: Side of triangle.

The opposite is the m side. The adjacent is the n side. The hypotenuse is o side.
Writing down suggestions for Putri to improve her understanding. In your suggestions, make sure, there is an explanation.

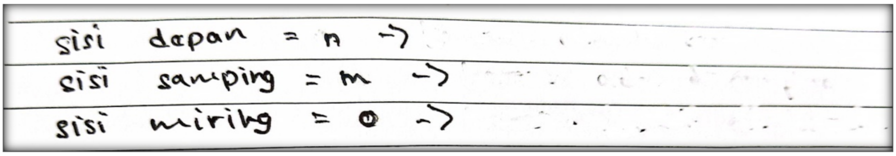
This question aims to test the respondents' understanding of correcting a mistake. Students were given an invalid solution and asked to identify and correct the error by applying their knowledge of right triangle side relatuionships. The following results were presented from the selected respondents' work.


sisi depan adalah o
sisi samping adalah n
sisi miring (hipotenusa) adalah m
karena pada dasarnya dimana arah sudut yg ada pada segitiga
itu lah sisi depannya.

Remark: The front side is o, The side is n, slant side (hypotenuse) is m.

Figure 4: Respondent 3's solution for question 2.

According to Figure 4, Respondent 3 (R3) employed the narrative argument to answer the question. The statement *"di mana arah sudut yang ada pada segitiga itulah sisi depannya"* (where is the direction of the angle in the triangle then that is the opposite side) suggested that the respondent knew the concepts of "opposite side," "adjacent side," and "hypotenuse" in the context of right triangles. Nevertheless, the respondent encountered difficulties in adequately implementing it. This was because the respondent possessed only knowledge without a comprehensive grasp of it despite these terms being presented in class. Therefore, the student struggled with demonstrating reasoning when the statement given was incorrect and what the correct one should be.



Remark: The front side = n, the side = m, the slant side = 0 .

Figure 5: Respondent 4's solution for question 2.

Figure 5 explains that students did not present the step of identifying and correcting the mistakes in the statement given, although the question already emphasises it.

Students faced difficulty demonstrating reasoning and proof when correcting mistakes in a given mathematical statement. They often struggle to identify errors in the provided mathematical statements. The observed challenge in rectifying errors stemmed from insufficient conceptual comprehension. Students encountered difficulties comprehending the fundamental principles or concepts pertaining to the statement, making it arduous for them to recognise and correct inaccuracies. Some students bypassed the reasoning process and asserted or concluded the correct information. This indicates a possible deficiency in their ability to explain modifications. Simply providing the correct answer without demonstrating the reasoning limits their comprehension. In this example, failure to justify the relationship among the sides of triangles may result in inconsistency in using the Pythagorean theorem. In practice, the longest side, which is called the hypotenuse, should be located opposite the right angle in the triangle.

Question 3: In the two pictures below, compare the number of tables and chairs. A rectangular table with four chairs is available. When two tables are joined together, they can hold six chairs, as illustrated in the Figure 6 below.

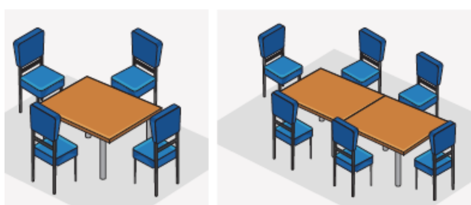
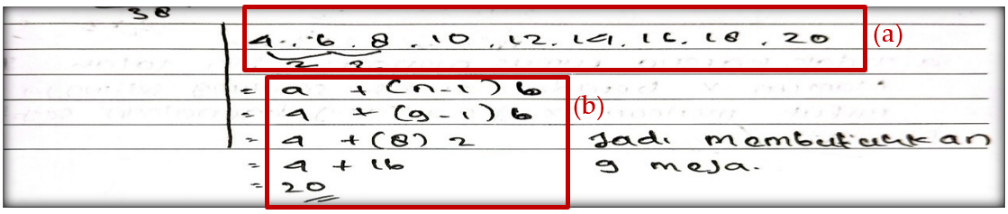


Figure 6: Rectangular tables with chairs.

How many tables are required if 20 persons eat together at one table? How did you find out? Explain your answer.

This question falls under the conjecture making category. Students needed to determine the relationship between people and table using given chair and table figures. They could then identify a rule explaining this relationship and apply it to calculate how many tables would be needed for 20 people. Finally, they developed a generic rule for calculating tables needed for any number of people. Thus, students demonstrated decision making by planning and implementing strategies based on their reasoning to develop solutions.

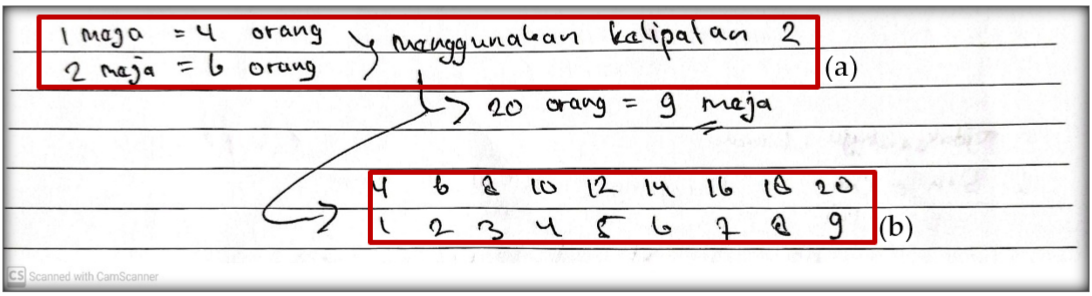
The following results were presented from the selected respondents' work.



Remark: Thus, 9 is proved.

Figure 7: Respondent 2's solution for question 3.

Figure 7 depicts R2's responses. The respondent calculated the answer manually, as shown in (a). In addition, it shows that the respondent used the concept of sequence to find the answer, as shown in (b); however, the respondent was unable to build a relationship with the concept. Therefore, the respondent misapplied the concept. The respondent only used the concept to confirm the answer, which was found by counting manually first. As a result, the respondent failed to find the answer by identifying a pattern to make a conjecture.



Remark: 1 table = 4 people; 2 table = 6 people

Figure 8: Respondent 4's solution for question 3.

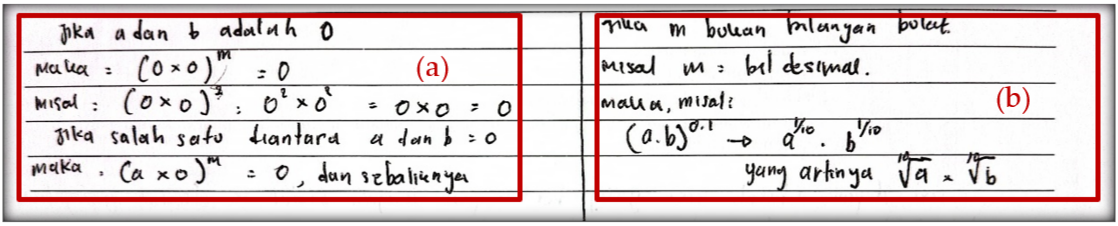
Figure 8 displays the respondents' answers to question 3. It shows that a student engaged in inductive reasoning by identifying the pattern and finding that the number of people is "$2n$" as the respondent stated, *"menggunakan kelipatan 2."* (using multiples of 2). However, instead of using the pattern, students resorted to manual listing to count the number of tables for each chair addition, reflecting errors in pattern recognition (Figure 8). Ideally, students should derived the general pattern $p = 2n + 2$, where n is the number of tables and p is the number of people (chairs).

In conclusion, the analysis revealed students' consistent difficulty in identifying patterns to form conjectures. Most notable, they replied heavily on basic techniques like counting, drawing and repeated addition. Although these methods do yield correct answers, many students still struggle with inductive reasoning, particular in formulating mathematics conjectures.

Question 6: Prove the following exponent property:

$$(ab)^m = a^m \times b^m, \quad \text{where} \quad a, b \neq 0 \text{ and } m \text{ is an integer.}$$

This question includes developing an argument category. Students were assigned to explain why the statement given is held. The following are students' responses to solving reasoning and proof questions in developing an argument.



Jika a dan b adalah 0
Maka : $(0 \times 0)^m = 0$
Misal : $(0 \times 0)^2 = 0^2 \times 0^2 = 0 \times 0 = 0$
Jika salah satu diantara a dan b = 0
maka . $(a \times 0)^m = 0$, dan sebaliknya

(a)

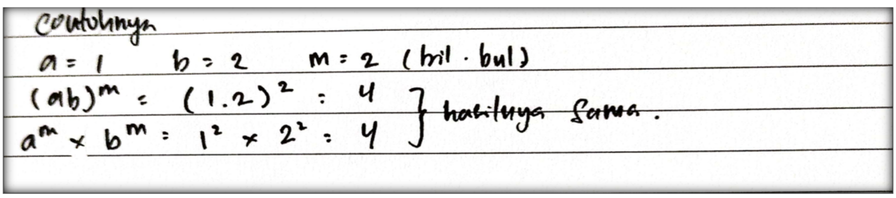
Jika m bukan integer boleh
Misal m : bil desimal.
maka, misal:
$(a.b)^{0.1} \to a^{1/10} . b^{1/10}$
yang artinya $\sqrt[10]{a} \times \sqrt[10]{b}$

(b)

Remark: If a and b are 0 thus $(0 \times 0)^m = 0$
Example: $(0 \times 0)^2 = 0^2 \times 0^2 = 0 \times 0 = 0$.
If one of a or b is 0, thus $(0 \times b)^m = 0$, and vice versa

Figure 9: Respondent 1's solution for question 6.

Figure 9 shows R1's response to question 6: In displaying the answer, the respondent divided it into two parts, namely "if $a, b = 0$", as shown in (a), and "if m is not an integer", as shown in (b), which indicates that the respondent develops the argument and focuses on giving a counterexample on prerequisites ($a, b \neq 0$ and m is an integer), where if $a, b = 0$ and if m is not an integer. However, in this case, the respondent's description of the process of reasoning and proof does not include any conclusions as to why the statement holds.



Contohnya
a = 1 b = 2 m = 2 (bil . bul)
$(ab)^m = (1.2)^2 = 4$
$a^m \times b^m = 1^2 \times 2^2 = 4$ } hasilnya sama.

Remark: $(ab)^m = (1.2)^2 = 4$
$a^m \times b^m = 1^m \times 2^m = 4$ — same outcomes

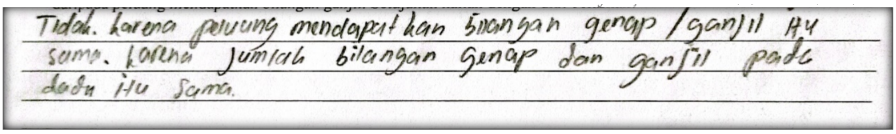
Figure 10: Respondent 6's solution for question 6.

Figure 10 shows the respondent's (R6) responses to question 6, all the respondents used examples to develop an argument. Therefore, the respondent did not explain why the property was held; instead, they just implemented it.

A prevalent issue among the respondents was the inclination to construct arguments based on examples. This strategy has the potential to result in proofs that are not fully comprehensive, and it highlights the challenge of extending arguments beyond specific cases.

Question 7: Your friend told you that when you multiply two dice, the chances of getting an even number are higher than an odd number. Do you agree? Explain it.

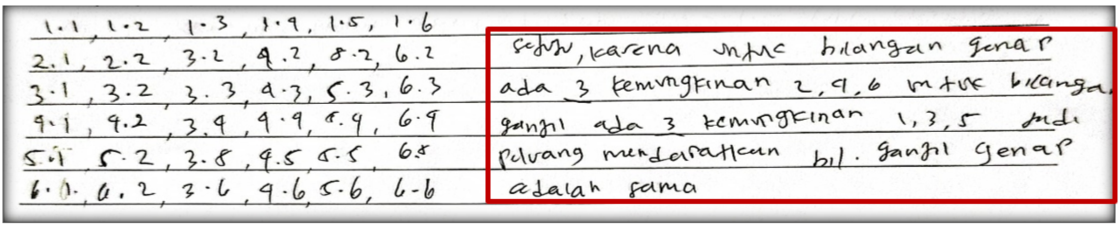
This question includes evaluating an argument category. Students were assigned to evaluate whether the justification is correct.


Tidak. karena peluang mendapatkan bilangan genap / ganjil itu sama. karena jumlah bilangan genap dan ganjil pada dadu itu sama.

Remark: No. Because chances of getting even / odd numbers are the same. Number of even and odd in the data are the same.

Figure 11: Respondent 3's solution for question 7.

Based on R3's response as shown in Figure 11, the respondent did not understand the questions well. The respondent gave the wrong condition, hence providing the wrong conclusion. The respondent stated that the probability of getting the even and odd numbers is the same since the number of even and odd numbers in dice is equal. Hence, by communicating within the context, students are able to define the situation logically.


1.1, 1.2, 1.3, 1.4, 1.5, 1.6
2.1, 2.2, 3.2, 4.2, 5.2, 6.2
3.1, 3.2, 3.3, 4.3, 5.3, 6.3
4.1, 4.2, 3.4, 4.4, 5.4, 6.4
5.1, 5.2, 3.5, 4.5, 5.5, 6.5
6.1, 6.2, 3.6, 4.6, 5.6, 6.6

setuju, karena untuk bilangan genap ada 3 kemungkinan 2, 4, 6 untuk bilangan ganjil ada 3 kemungkinan 1, 3, 5 jadi peluang mendapatkan bil. ganjil genap adalah sama

Remark: Agree because the number of even is 3, possible 2, 9, 6 for the number odd is 3; possible 1, 3, 5. So, chances getting odd even are the same.

Figure 12: Respondent 5's solution for question 7.

Figure 12 displays R5's response, which shows the respondent's systematic work to evaluate the argument's validity. Starting by writing down all possibilities, the respondent failed to conclude. It indicated that the respondent could not build a relationship between the condition and the conclusion.

In conclusion, students faced difficulty demonstrating the reasoning and proof needed to evaluate an argument category. Many students struggle to grasp the underlying mathematical concepts. Certain students may struggle to comprehend the fundamental principles that underline the mathematical argument they need to evaluate. Furthermore, students encountered difficulties establishing connections between conditions, premises, and conclusions.

5 Discussion

The findings indicate that students' abilities to demonstrate reasoning and proof remains inadequate as depicted by the low average score of 21.33%. This poor performance reflects significant difficulties across key aspects of mathematical reasoning. Students struggle particularly in making conjectures, justifying mathematical statements, evaluating arguments, and formulating coherent explanations. While qualitative analysis revealed that students employed various strategies-including using counterexamples, applying algebraic operations, and identifying patterns-these approaches were often undermined by their failure to integrate fundamental mathematical concepts. This disparity highlights the urgent need to incorporate reasoning and proof skills more systematically.

This study delineates the specific obstacles encountered by secondary school students, with a paramount concern being their inclination to use specific examples as proof. This is consistent with the results of prior investigations by [48, 14].

The findings show students often validate their claims through examples while failing to understand the limitations of this approach. The issue lies in the students' potential lack of understanding regarding the limitations and constraints of relying on supporting examples for reasoning. As stated by [43], an ongoing observation globally is that students often depend on supported examples as proof without fully recognising the constraints of such reasoning. Put another way; students often rely on specific examples to support their arguments without fully grasping the underlying concepts or the broader relevance of their reasoning. This indicates a targeted education intervention that develop comprehensive reasoning skills beyond example-based arguments through deeper conceptual understanding.

The findings revealed students limited cognitive skills due to their over reliance on their prior knowledge with contrained reasoning. This align with [26]'s finding in demonstration of how statistical reasoning and prior mathematical knowledge affect achievement. While focused on statistics, this depicts the importance of learning effort for self-challenge. Thus, mathematical understanding requires comprehension of formulae, enquiries, and symbols, while educators need core competencies to guide students effectively [5]. Additionally, observing students' cognitive frameworks aligns with Shulman's framework of pedagogical content knowledge.

In addition, the findings revealed that students exhibit a deficiency in comprehending specific concepts. A study by also [15] found this observation consistent. Based on this study, it has been found that high school students face challenges in their ability to reason and prove, mainly because they lack a thorough understanding of fundamental concepts, definitions, and notation in the subject area. Essentially, symbolic representations such as notation play a role in communicating mathematical concepts with reasoning [33]. The findings of this study suggest that the essential mathematical concepts expressed through notation and definition should not be overlooked. In a discourse, the emphasis on developing concepts through definition and notation is critical to assisting students to go further into application.

One significant obstacle that has been observed is the student's lack of ability to link previously learn concepts. This, to a large extent, hinder their ability in reasoning and proof skills as similarly highlighted by [49]. Thus, establishing conceptual connections via mathematical process is vital. Inquiry based approaches are particularly effective, forstering connections and reasoning via counterexamples and diagrams [18, 17]. To enhance problem-solving skills, we strongly recommend educators consistently implement these processes as outlined by the mathematics education framework.

Moreover, the discussion identifies a recurring issue with students' demonstration of reasoning and proof skills. The research highlighted that students frequently neglect the final stage of their proofs. For instance, [29] revealed that students committed minor errors by omitting proper conclusion steps. This pattern suggests difficulties both in initiating reasoning and ineffectively finalising arguments.

An key discovery was the difficulty faced by students in applying mathematical knowledge for reasoning and proof. This finding aligns with the results of [12], where many students avoided questions on proof, stating "I am uncertain about how to initiate the proof / *"Saya tidak yakin* *tentang bagaimana memulai pembuktian."* This suggests a more profound difficulty in converting abstract information into tangible problem-solving abilities.

The inadequate emphasis on reasoning and proof in current mathematics classroom practices tends to directly contributes to students' struggle. As highlighted by [44, 43], students focus disproportionately on practicing and memorising procedures rather than developing proof skills. This limited exposure restricts opportunities to cultivate these essential abilities. Hence, the findings offer critical insights into students' cognitive processes underlying reasoning and proof that are crucial for performance in assessment like PISA. This study emphasizes the necessity of enhancing cognitive skills through curriculum and instructional that prioritise mathematical process, definitions and notations to achieve advanced mathematical proficiency.

6 Conclusions

This mix-method study investigated how students demonstrate reasoning and proof in mathematical problem-solving. The findings show three key issues:

(1) Students struggle significantly with reasoning and proof, often depending on single examples for validation.

(2) Their lack of conceptual understanding directly contributes to low performance.

(3) Most fail to establish logical connections between premises and conclusions.

These findings suggest that teachers should prioritise mathematical processes like making connections, employing representation, using counterexamples, and implementing problem-solving strategies over repetitive practice. Furthermore, research must identify learning environments that can effectively develop students' mathematical reasoning and proof ability.

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